Analytic theory

Historical tour

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Statistical theory

Universality of wave growth

The theory for wave forecasting

Conclusions

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Analytic theory of wind-driven seas

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Outlines

1. Sea wave forecasting history
2. Hamiltonian equations for water waves and the kinetic equation
3. Statistical description of wind waves
4. Universality of wind wave growth
5. The theory for wave forecasting
6. Conclusions
7. References
Old is always gold

The Beaufort Scale is an empirical measure for describing wind speed based mainly on observed sea conditions.

Its full name is the Beaufort Wind Force Scale.

Sir Francis Beaufort (1774-1857)
Ivan Ivazovsky. *Brig Mercury attacked by two Turkish ships (1892).* Beaufort force 6
Ivan Ivazovsky. *The ninth wave (1850)*
Beaufort force 12
Weather and wave forecasting

Urbain Jean Joseph Le Verrier  
(1811-1877)

Robert FitzRoy  
(1805-1865)

Meteo-France, 1854

Met Office, UK, 1854

Both are created just after the Great Storm 14 November 1854 (the Crimean War). Anti-Russian coalition lost 1500 men, 53 ships incl. 25 cargos, goods and ammunition for 60 mln. francs.
The old is always new
The World War II and the concept of significant wave height

In particular, by studies for the Overlord operation (1944)

Harald Ulrik Sverdrup 1888-1957

‘The concept of ‘significant waves’ is essential for purpose of forecasting’

Walter Heinrich Munk 1917
The theory

\[
\eta(r, t) - \text{surface elevation}
\]

\[
r = (x, y) - \text{horizontal coordinates}
\]

\[
\text{div} \mathbf{V} = 0; \quad \mathbf{V} = \nabla \Phi; \quad \Delta \Phi = 0
\]

\[
\frac{\partial \eta}{\partial t} = \frac{\delta H}{\delta \psi}; \quad \frac{\partial \psi}{\partial t} = -\frac{\delta H}{\delta \eta}; \quad H = T + U - \text{total energy}
\]

\[
U = \frac{1}{2} \int \eta^2 \, d\mathbf{r} - \text{potential}; \quad H = \frac{1}{2} \int \psi \frac{\partial \psi}{\partial \mathbf{n}} \, dS - \text{kinetic}
\]
Theory of weakly nonlinear waves

\[ T = \frac{1}{2} \int_r \int_{-\infty}^{\eta} (\nabla \Phi)^2 \, dz = - \int G(s, s') \Psi(s) \Psi(s') \, dsds' \]

\[ G(s, s') = G(s', s) \] – the Green function of
the Dirichlet-Neuman problem

\[ \Phi = \Psi \bigg|_{z=h}; \quad \frac{\partial \Psi}{\partial z} \to 0, \quad z \to \infty \]

\[ H = H_0 + H_1 + H_2 + \ldots; \quad \mu \approx k \eta \] – average steepness
\[ \mu^2 + \mu^3 + \mu^4 \]
Hamiltonian equations and normal variables

We use non-symmetric Fourier transform

\[ \eta(r, t) = \int \eta(k) \exp(ikr) dk; \quad \Psi(r, t) = \int \Psi(k) \exp(ikr) dk \]

Normal variables

\[ \eta_k = \left( \frac{\omega_k}{2g} \right)^{1/2} (a_k + a_{-k}^*); \quad \Psi_k = \left( \frac{g}{2\omega_k} \right)^{1/2} (a_k - a_{-k}^*) \]

\[ H = \int \omega_k a_k a_k^* dk + H_{int}; \quad \omega_k = \sqrt{g|k|} - \text{dispersion law} \]

\[ \frac{\partial a_k}{\partial t} = i \frac{\delta \tilde{H}}{\delta a_k^*}; \quad \tilde{H} = \frac{1}{4\pi^2} H \]

\[ \langle a_k a_k^* \rangle = N_k \delta(k - k'); \quad \epsilon_k = \omega_k N_k \]
The most important nonlinear interactions

4-wave resonances

\[
\begin{align*}
\omega_0 + \omega_1 &= \omega_4 + \omega_3 \\
\mathbf{k}_0 + \mathbf{k}_1 &= \mathbf{k}_4 + \mathbf{k}_3
\end{align*}
\]

Phillips’ curve

\[
\begin{align*}
2\omega_0 &= \omega_4 + \omega_3 \\
2\mathbf{k}_0 &= \mathbf{k}_4 + \mathbf{k}_3
\end{align*}
\]
Owen Martin Phillips (1930–2010)
Canonical transformation – eliminating three-wave interactions – the effective Hamiltonian

Canonical transformation of normal variables

$$a_k \rightarrow b_k$$

$$H \Rightarrow \int \omega_k b_k b_k^* + \frac{1}{2} \int T_{k,k_1,k_2,k_3} b_k^* b_{k_1}^* b_{k_2} b_{k_3} \delta_{k+k_1-k_2-k_3} \, d k k_1 k_2 k_3$$

$$T(\varepsilon k, \varepsilon k_1, \varepsilon k_2, \varepsilon k_3) = \varepsilon^3 T(k, k_1, k_2, k_3)$$
Matrix element $T$ (we put $g = 1$ here)

\[
\tilde{T}_{k_1 k_2 k_3 k_4} = -\frac{1}{4} \frac{1}{(k_1 k_2 k_3 k_4)^{1/4}} \left\{ 
\begin{array}{l}
+ \frac{1}{2} \left( k_{1+2}^2 - (\omega_1 + \omega_2)^4 \right) \left[ (k_1 k_2 - k_1 k_2) + (k_3 k_4 - k_3 k_4) \right] \\
- \frac{1}{2} \left( k_{1-3}^2 - (\omega_1 - \omega_3)^4 \right) \left[ (k_1 k_3 + k_1 k_3) + (k_2 k_4 + k_2 k_4) \right] \\
- \frac{1}{2} \left( k_{1-4}^2 - (\omega_1 - \omega_4)^4 \right) \left[ (k_1 k_4 + k_1 k_4) + (k_2 k_3 + k_2 k_3) \right] \\
+ \left( \frac{4(\omega_1 + \omega_2)^2}{k_{1+2} - (\omega_1 + \omega_2)^2} - 1 \right) (k_1 k_2 - k_1 k_2)(k_3 k_4 - k_3 k_4) \\
+ \left( \frac{4(\omega_1 - \omega_3)^2}{k_{1-3} - (\omega_1 - \omega_3)^2} - 1 \right) (k_1 k_3 + k_1 k_3)(k_2 k_4 + k_2 k_4) \\
+ \left( \frac{4(\omega_1 - \omega_4)^2}{k_{1-4} - (\omega_1 - \omega_4)^2} - 1 \right) (k_1 k_4 + k_1 k_4)(k_2 k_3 + k_2 k_3) 
\end{array} \right\}
\]
We need in statistical description

\[ \langle b_k b_k^* \rangle = N_k \delta_{k-k'}; \quad -N_k - \text{spectrum of wave energy satisfying the Hasselmann equation} \]

\[
\frac{dN_k}{dt} = S_{nl}
\]

\[
S_{nl} = \pi g^2 \int |T_{kk_1k_2k_3}|^2 dkdk_1dk_2dk_3
\]

\[
(N_k N_{k_1} N_{k_2} + N_k N_{k_1} N_{k_4} - N_k N_{k_2} N_{k_3} - N_{k_1} N_{k_2} N_{k_3})
\]

\[
\times \delta(k + k_1 - k_2 - k_3)\delta(\omega + \omega_1 - \omega_2 - \omega_3)
\]

\[
\omega = \sqrt{gk} - \text{dispersion law}
\]

Energy spectrum (real energy spectrum is \( \rho_w g \varepsilon_k \))

\[
- \varepsilon_k = \omega_k N_k
\]

Spectrum of surface elevation

\[
- I_k = \langle |\eta_k|^2 \rangle = \frac{1}{2} \omega_k (N_k + N_{-k})
\]
Klaus Hasselmann (JFM, 1962)
What is the energy spectrum?

\[ \varepsilon(k) = \omega_k N_k \]

\[ \varepsilon(k) dk = \frac{2 \omega^3}{g^2} \varepsilon(\omega, \theta) d\omega d\theta \]

\[ \varepsilon(\omega) = \frac{1}{2\pi} \int_0^{2\pi} \varepsilon(\omega, \theta) d\theta \text{ - omnidirectional spectrum} \]

\[ \int \varepsilon(\omega) d\omega = \int \varepsilon(f) df = \langle \eta^2 \rangle = \sigma^2 \text{ - variance} \]

With a good accuracy the amplitude distribution is gaussian

\[ P(\eta) = \frac{1}{\sqrt{2\pi}\sigma} \exp\left(-\frac{\eta^2}{2\sigma^2}\right) \]
the Rayleigh distribution

\[ P(H) = \frac{H}{4\sigma^2} \exp\left(-\frac{H^2}{8\sigma^2}\right) \]

\[ \langle H \rangle = \sqrt{2\pi}\sigma \approx 2.5\sigma \]

\[ H_{\text{rms}} = \langle H^2 \rangle^{1/2} = 2\sqrt{2}\sigma \approx 2.83\sigma \]

\[ H_s - \text{significant height}; \]

\[ H_s = 4\sigma \quad \text{could be 30 m or even more} \]

\[ H_{1/10} \approx 5.09\sigma \]

\[ H_{\text{rogue}} > 2.1H_s > 8.4\sigma \]

In a strong storm \( H_s \approx 0.22 \frac{U^2}{g} < 14 \text{ m} - \text{non-gaussian PDF} \]
From the concept of significant wave height to the spectral theory of wind seas (Gelci et al. 1957)

Wave spectra - decomposition in wave scales (frequency or wavenumbers)

\[ E = \int_0^\infty E(f)df \]

Fig. 25. Evolution of wave spectra with fetch for offshore winds (11-12 h, Sept. 15, 1968). The spectra are labelled with the fetch in kilometres. (From Hasselmann et al., 1973.)
More ideas on universal shaping (similarity) of wave spectra

The spectra do not depend (depend slightly) on conditions of wave spectra evolution

Donelan et al. 1985
More ideas on spectral shapes in log-axes

O. Phillips, 1985

P. Liu, 1989

Hwang & Wang, 2004
Figure 4a. The spectrum of surface elevation for the case of recent swell. The $-4$ power law fitted to Figure 4b is also shown here.

Figure 4b. The spectrum of surface elevation for the strongly wind-generated case. Above the peak, the spectrum conforms to a $-4$ power law (Donelan et al., 1985).
Spectral tail

Numerous experiments performed in wave tanks as well as field observations show that the tail of the omnidirectional wave energy spectrum

\[ E(\omega) \sim \frac{\alpha g U}{\omega^4}; \quad \alpha - \text{dimensionless constant} \]

Resio, Long & Vincent (2004) offer more accurate formula

\[ E(\omega) \sim \frac{\alpha_4 g (U^2 C_p)^{1/3}}{\omega^4}; \quad F(k) \sim \beta k^{-5/2} \]

\( C_p \) – group velocity of the spectral maximum

\[ \alpha_4 \approx 5.5 \cdot 10^{-3} \]
Wave spectra scaling by energy flux

For \( F(k) = \beta k^{-5/2} \iff \varepsilon = C_k \frac{P^{1/3}}{\omega^4} \)
Wave growth and wind speed scaling

Following Kitaigorodskii (1962) we introduce dimensionless quantities

\[ \chi = xg/U_{10}^2 \]
\[ \tilde{\varepsilon} = \sigma^2g^2/U_{10}^4 \]
\[ \tilde{\omega} = \omega U_{10}/g \]

\( \omega_p \) – frequency of the spectral peak

\( U_{10} \) – wind speed at the ‘standard anemometer’ height 10 meters

Experiments show that in the fetch-limited conditions (spectrum does not depend on time) \( \tilde{\varepsilon} \) and \( \tilde{\omega} \) are power-like functions

\[ \tilde{\varepsilon} = \tilde{\varepsilon}_0 \chi^p \]
\[ \tilde{\omega} = \tilde{\omega}_0 \chi^{-q} \]

0.7 < \( p \) < 1.1; 0.68 < \( 10^7 \tilde{\varepsilon}_0 \) < 18.6
0.23 < \( q \) < 0.33; 10.4 < \( \tilde{\omega}_0 \) < 22.6
Figure 5.4: A composite of data from a variety of studies showing the development of the non-dimensional energy, $\varepsilon$, as a function of non-dimensional fetch, $\chi$. The original JONSWAP study (Hasselmann et al., 1973) used the data marked JONSWAP, together with that of Burling (1959) and Mitsuyasu (1968). Also shown are a number of growth curves obtained from the various data sets. These include: JONSWAP Eq.(5.27), Donelan et al (1985) Eq.(5.33) and Dobson et al (1989) Eq.(5.38).
### Exponents of fetch-limited growth in experiments

<table>
<thead>
<tr>
<th>Experiment</th>
<th>$p_X$</th>
<th>$q_X$</th>
<th>$z_X$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Black Sea</td>
<td>0.89</td>
<td>0.275</td>
<td>0.010</td>
</tr>
<tr>
<td>Walsh et al. 1989, US coast</td>
<td>1.0</td>
<td>0.29</td>
<td>0.033</td>
</tr>
<tr>
<td>Kahma &amp; Calkoen, 1992, unstable</td>
<td>0.94</td>
<td>0.28</td>
<td>0.027</td>
</tr>
<tr>
<td>Kahma &amp; Calkoen 1992, stable</td>
<td>0.76</td>
<td>0.24</td>
<td>0.040</td>
</tr>
<tr>
<td>Kahma &amp; Pettersson 1994</td>
<td>0.93</td>
<td>0.28</td>
<td>0.020</td>
</tr>
<tr>
<td>Davidan 1980</td>
<td>1.0</td>
<td>0.28</td>
<td>0.067</td>
</tr>
<tr>
<td>JONSWAP by Phillips, 1977</td>
<td>1.0</td>
<td>0.25</td>
<td>0.167</td>
</tr>
<tr>
<td>Kahma &amp; Calkoen, 1992, composite</td>
<td>0.9</td>
<td>0.27</td>
<td>0.033</td>
</tr>
<tr>
<td>Kahma 1981, 1986, rapid growth</td>
<td>1.0</td>
<td>0.33</td>
<td>-0.100</td>
</tr>
<tr>
<td>Kahma 1986, average growth</td>
<td>1.0</td>
<td>0.33</td>
<td>-0.100</td>
</tr>
<tr>
<td>Donelan et al, 1992, St. Claire</td>
<td>1.0</td>
<td>0.33</td>
<td>0.023</td>
</tr>
<tr>
<td>Ross, 1978 unstable, Michigan</td>
<td>1.1</td>
<td>0.27</td>
<td>0.167</td>
</tr>
<tr>
<td>Liu &amp; Ross, 1980, stratification</td>
<td>1.1</td>
<td>0.27</td>
<td>0.167</td>
</tr>
<tr>
<td>JONSWAP Hasselmann et al.,1973</td>
<td>1.0</td>
<td>0.33</td>
<td>-0.010</td>
</tr>
<tr>
<td>Mitsuyasu, 1971</td>
<td>1.008</td>
<td>0.33</td>
<td>-0.095</td>
</tr>
<tr>
<td>ZRP numerics</td>
<td>1.0</td>
<td>0.3</td>
<td>0</td>
</tr>
</tbody>
</table>

$$z_X = \frac{2p_X - 10q_X}{3}$$ — “magic relation” mismatch

Adopted from Badulin, Babanin, Zakharov, Resio 2007
Features of self-similarity of wave spectra evolution results in simple link between exponent of energy growth $p$ and one of frequency downshift $q$.

$$p = \frac{10q - 1}{2}$$
Universality of wind-wave growth

‘Magic links’ and spectral features of wind-waves can be summarized in a remarkably simple wind-free algebraic relationship

\[ \mu^4 \nu = \alpha_0^3 \]

\[ \mu = \langle ak_p \rangle \quad \text{– wave steepness} \]

\[ \nu = \omega_p t (2k_p x) \quad \text{– number of waves} \]

\[ \alpha_0 \approx 0.7 \quad \text{– a universal constant} \]
Our wind-free invariant in sea experiments

\[ \tilde{H} = \frac{H_s}{\text{Fetch}}; \quad \tilde{T} = T \sqrt{\frac{g}{(8\pi^2 \text{Fetch})}} \]

\[ \tilde{H} \sim \tilde{T}^{5/2} \]

Waves in a sector \( \pm 30^\circ \) to the off-shore direction for up to 15 years in the US nearshore.
Theoretical explanation

Go back to the resonant conditions

\[ k_1 + k_2 = k_3 + k_4 \]
\[ \omega_1 + \omega_2 = \omega_3 + \omega_4 \]

Long-short interaction, asymptotics If \( k_1 \ll k_2, k_3 \ll k_4 \) then

\[ k_2 \approx k_4 \]
\[ |k_1| \approx |k_3| \]
Kernel $T$ asymptotics

$$T_{k,k_1,k_2,k_3} = -\frac{1}{2} |k|^2 \left| k_2 \right| T_{\theta_1\theta_2}$$

$$T_{\theta_1\theta_2} = (\cos \theta_1 + \cos \theta_3) \cos(\theta_1 - \theta_3)$$

$\theta_1$ and $\theta_2$ are the angles between the small vectors and the base vector.
This small parameter expansion has two first orders cancelled.
As a result, the long-short interactions are weak.
Let us consider the stationary Hasselmann equation

\[ S_{nl} = 0 \]

The ‘naive’ solutions

\[ N_k = \frac{T}{\omega_k + \mu} \]

are not solutions because of integral divergence.

The wind-driven sea is very far from the thermodynamic equilibrium.

Similar to turbulence in incompressible fluid in the isotropic case the Hasselmann equation has two conservation laws:

- energy \( \varepsilon \)
- wave action \( N \)

\[ \varepsilon = \int \omega_k N_k \, dk; \quad N = \int N_k \, dk \]
Isotropic power-like solutions

Let $N = k^{-x}$. Then

$$S_{nl} = g^3 k^{-3x} + \frac{19}{2} F(x)$$

$F(x)$ is defined for $\frac{5}{2} < x < \frac{19}{4}$

$$F \rightarrow \frac{5 \times 5 \times 2^{-3} \times \pi^3}{x - \frac{5}{2}}, \quad x \rightarrow \frac{5}{2} \quad \text{and}$$

$$F \rightarrow \frac{5 \times 11 \times 19 \times 2^{-13} \times \pi^3}{\frac{19}{4} - x} \approx , \quad x \rightarrow \frac{19}{4}$$
Kolmogorov spectra

There are two and only two solutions!

\[ N_k^{(1)} = c_p \left( \frac{P_0}{g^2} \right)^{1/3} \frac{1}{k^4}, \]

\[ N_k^{(2)} = c_q \left( \frac{Q_0}{g^{3/2}} \right)^{1/3} \frac{1}{k^{23/6}}. \]

\[ c_p = \left( \frac{3}{2\pi F'(4)} \right)^{1/3}, \quad c_q = \left( \frac{3}{2\pi |F'(23/6)|} \right)^{1/3} \]

The calculation yields:

\[ c_p = 0.203 \quad c_q = 0.194 \]
Corresponding energy spectra are

\[
\varepsilon^{(1)}(\omega) = \frac{4\pi c_p g}{\omega^4} (gP_0)^{1/3} \quad (*)
\]

\[
\varepsilon^{(1)}(\omega) = \frac{4\pi c_q g}{\omega^{11/3}} (gQ_0)^{1/3}
\]

Here \( P_0 \simeq \sigma^2/\tau \) – flux of the oceanographic energy \( (gP_0)^{1/3} \) has the dimension of velocity.

The real energy flux from air to water

\[ P \simeq \rho_a g P_0 \simeq \rho_a u^3 \]

\( Q \) is the flux of wave action to large scales.

**Spectrum** \( \varepsilon^{(1)}(\omega) \) explains numerous experiments
Fetch-limited growth

The stationary Hasselmann equation

\[ \frac{\cos \theta}{2\omega} \frac{\partial \varepsilon}{\partial x} = S_{nl} \]

has self-similar solutions

\[ \varepsilon(\omega, \theta) = \beta^b \chi^{p+q} \Phi(\beta \omega \chi^q, \theta) \]

\(\beta - a\) constant, \(p, q\) are connected with the ‘magic relation’

\[ 10q - 2p = 1 \]

Total energy \(\varepsilon = \beta^5 \chi^p E_0\), peak frequency \(\omega_p = \frac{\omega_0}{\beta} \chi^{-q}\)

Universality parameters do not depend on \(\beta\) and \(p\) !!!

\[ \mu^4 k_p x \sim \varepsilon^2 \omega_p^{10} x = E_0^2 \omega_0^{10} \]

Self-similarity explains both magic relation and universality
Swell within the kinetic equation
Fetch-limited setup

The total wave action

\[ N = \int \frac{\varepsilon(\omega)}{\omega} d\omega \simeq \beta^4 \chi^{p+q} \]

For the stationary swell \( N = \text{const} \)

\[ p + q = 0; \quad q = \frac{1}{12}; \quad p = -\frac{1}{12} \]

The swell is described by the solution

\[ \varepsilon(\omega, \theta, \chi) \simeq \beta \Phi(\beta \omega \chi^{1/12}, \theta) \]

Total energy decreases as \( \chi^{-1/12} \) due to loss of energy due to rare wave breaking
Swell in a homogeneous ocean
Duration-limited setup

\[ \frac{\partial \varepsilon}{\partial t} = S_{nl} \]

has a family of self-similar solutions

\[ \varepsilon(\omega, \theta, \tau) = \beta^{11/2} \tau^{p+q} \Phi(\beta \omega \tau^q, \theta) \]

\[ \tau = tg/U \text{ – dimensionless duration.} \]

The magic condition

\[ 9q - 2p = 1 \]

Universality conditions holds in all numerical experiments.

The homogeneous swell is described by solution

\[ \varepsilon(\omega, \theta, \tau) \simeq \beta^{11/2} \Phi(\beta \omega \tau^{1/11}, \theta) \]

For the homogeneous swell \( N = \text{const} \)

\[ p + q = 0; \quad q = \frac{1}{11}; \quad p = -\frac{1}{11} \]
Easy to get in simulations

\[ n = t^{2/11} U_0 (\omega^2 t^{1/11}, \Theta) \]

Swell

\[ \omega_1 \]

\[ \log_{10} (U(\xi)) \]

\[ \xi = \log_{10} (\omega_t^{2/3}) \]
Growing wind sea.
Self-similarity in an explicit form
In the wind-driven sea the conservative Hasselmann equation must be replaced by more realistic equation

\[ \frac{\partial \varepsilon}{\partial t} + \frac{\partial \omega_k}{\partial k} \frac{\partial \varepsilon}{\partial r} = S_{nl} + S_{in} + S_{diss} \]

- \( S_{nl} \) – derived from free surface Euler equations and known for sure;
- \( S_{in} \) – wind input term: multiple versions, local differences up to 500%;
- \( S_{diss} \) – wave dissipation: multiple LF and HF versions

Detailed discussion in Pushkarev & Zakharov 2016
The wind input term

\[ S_{in} = \gamma_k N_k \]

There are dozens versions of \( \gamma_k \). For instance,

- **Snyder et al. (1981)** forcing

  \[ \gamma = 0.2 \frac{\rho_a}{\rho_w} \omega \left( \frac{\omega}{\omega_0} \cos \theta - 1 \right) \]

  Widely used in prognostic models WAM-3, WAM-4, SWAN, WaveWatch III

- **Zakharov, Resio, Pushkarev (2012)** – ZRP forcing

  \[ S_{in}(\omega, \theta) = 0.05 \frac{\rho_{air}}{\rho_{water}} \omega \left( \frac{\omega}{\omega_0} \right)^{4/3} \epsilon(\omega, \theta) f(\theta) \]

  \[ f(\theta) = \begin{cases} 
  \cos^2 \theta, & \text{for } -\pi/2 < \theta < \pi/2 \\
  0, & \text{otherwise}
  \end{cases} \]

  \[ \omega_0 = \frac{g}{U_{10}} \]
Total energy exponent dependence on the fetch. Snyder vs ZRP forcing
Domination of the nonlinear wave interactions

Split into two terms

\[ S_{nl} = F_k - \Gamma_k \mathcal{N}_k \]  \hspace{1cm} (1)

where

\[ F_k = \pi g^2 \int |T_{0123}|^2 N_1 N_2 N_3 \]
\[ \times \delta(k + k_1 - k_2 - k_3)\delta(\omega_k + \omega_1 - \omega_2 - \omega_3)dk_1dk_2dk_3 \]  \hspace{1cm} (2)

\[ \Gamma_k = \pi g^2 \int |T_{0123}|^2 (N_1 N_2 + N_1 N_3 - N_2 N_3) \]
\[ \times \delta(k + k_1 - k_2 - k_3)\delta(\omega_k + \omega_1 - \omega_2 - \omega_3)dk_1dk_2dk_3 \]  \hspace{1cm} (3)
Domination of nonlinearity gives a chance for the analytical theory

The nonlinear relaxation rate is one (or more) orders higher than wind wave pumping rate!

Thus, an asymptotic model can be developed where effect of wave-wave resonant interactions is a dominating mechanism.
Today’s wave models

- SWAN – Simulating WAves Nearshore (Delft University of Technology)

Assume that in the balance equation

\[ S_{nl} + S_{in} + S_{diss} = 0 \]

all three terms has the same order
In reality

\[ S_{nl} = F_k - N_k \Gamma_k \text{ and} \]

\[ F_k \gg S_{in} \]

\[ F_k \gg S_{diss} \]

The main process is balance between \( F_k \) and \( N_k \Gamma_k \)

\[ \Gamma_k \gg \gamma_k \]
Wave breaking provides dissipation of wave energy in the spectral band $\omega > 3\omega_p$

In the energy-capacious wave range $\omega < 3\omega_p$ the dissipation is negligibly small.

The WAM model use the following dissipation term

$$\gamma_{diss} = C_D \hat{\sigma} \left( \frac{k}{\hat{k}} \right) \left( (1 - \delta) + \delta \left( \frac{k}{\hat{k}} \right) \right) \left( \frac{\hat{\alpha}}{\alpha_{PM}} \right)^p$$

$$\hat{\sigma} = (\sigma^{-1})^{-1}; \quad \hat{\alpha} = \frac{E\sigma^4}{g^2}$$

Default WAM values $C_D = 2.35 \times 10^{-5}; \quad p = 2; \quad \delta = 0$
The direct phase-resolving numerical experiments

A. Korotkevich & V. Zakharov
In the WAM model wrong dissipation is compensated by the wrong wind input term.

The result is good for limited dimensionless fetches $\chi \lesssim 10^3$. 
Conclusion

For developing better prognostic models we must optimize the wind input term and advance numerical code for solution of the exact Hasselmann equation.
References I


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References II


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Thank you